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Interesting relations between the elements of the three-channel scattering matrix

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Abstract. Formulae for the phase of each off-diagonal element in terms of the amplitudes of the diagonal elements of the 3×3 unitary symmetric S matrix, together with the inverse relations, are given. The formulae have been used to check numerical calculations by applying them to three coupled channel potential wells.

1. Introduction

In a recent paper (Kabir and Kermode 1987), we considered the various methods that have been proposed for parametrising the 2×2 submatrix in coupled channel nucleon-nucleon scattering when there are one or more inelastic channels. These methods involve an inelastic 2×2 real matrix N, which satisfies the subunitarity condition $1\geqslant\det(1-N^2)\geqslant0$. The off-diagonal element N_{12} satisfies the inequality $N_{12}\leqslant(1-|N_{11}+N_{22}|+N_{11}N_{22})^{1/2}=Q$, which can be represented geometrically by taking $x=N_{11}$, $y=N_{22}$ and $z=N_{12}$ (Kermode and Cooper 1985). An illustration of the surface $N_{12}=Q$ is given in the paper by Sprung (1985).

For the case of only one inelastic channel, $N_{12}=Q$ and N requires only two parameters. Thus, the 3×3 elastic S matrix is a special case (see also Sprung and Kermode 1982).

The unitary 3×3 S matrix is also particularly special because it requires six independent real parameters which can be constructed from appropriate combinations of the three real amplitudes and the three real phases of the three diagonal or three off-diagonal elements. The 4×4 S matrix, with ten parameters, has four real amplitudes (phases) for the diagonal elements and six real amplitudes (phases) for the off-diagonal elements. Thus, the choice of free parameters for the 4×4 case is not as wide as that for the 3×3 case. For example, consideration of the diagonal elements *alone* is not sufficient for a complete description of the four-channel scattering matrix (see Kermode 1967).

In this paper, we shall consider the 3×3 unitary S matrix and introduce interesting equations for the three off-diagonal phases in terms of the three diagonal amplitudes and vice versa.

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2. Method

The 3×3 unitary symmetric $\boldsymbol{S}$ matrix may be written in the form

$$\boldsymbol{S}=\begin{pmatrix} e^{i\delta_1} & 0 & 0 \\ o & e^{i\delta_2} & 0 \\ 0 & 0 & e^{i\delta_3}\end{pmatrix}\begin{pmatrix} \eta_1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & \eta_2 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & \eta_3\end{pmatrix}\begin{pmatrix} e^{i\delta_1} & 0 & 0 \\ 0 & e^{i\delta_2} & 0 \\ 0 & 0 & e^{i\delta_3}\end{pmatrix}$$

where $\gamma_{ij}=N_{ij}\exp(i\beta_{ij})$ and η_i is a real parameter. Thus

$$S_{ii}=\eta_i\exp(2i\delta_i) \tag{1}$$

and

$$S_{ij}=N_{ij}\exp[i(\delta_i+\delta_j+\beta_{ij})] \qquad i\neq j. \tag{2}$$

The unitarity condition $\boldsymbol{SS}^+=1$ gives

$$\sum_{j\neq i} N_{ij}^2+\eta_i^2=1 \qquad i=1,2,3 \tag{3}$$

and

$$N_{12}[\eta_1\exp(i\beta_{12})+\eta_2\exp(-i\beta_{12})]=N_{13}N_{23}\exp[i(\pi+\beta_{23}-\beta_{13})] \tag{4}$$

and cyclic $(1\to2\to3\to1)$ and complex conjugate. Squaring and adding the real and the imaginary parts of equation (4), we have

$$N_{12}^2(\eta_1+\eta_2)^2\cos^2\beta_{12}+N_{12}^2(\eta_1-\eta_2)^2\sin^2\beta_{12}=N_{13}^2N_{23}^2=N_{13}^2N_{23}^2(\cos^2\beta_{12}+\sin^2\beta_{12}).$$

Thus

$$\tan^2\beta_{12}=\frac{[N_{12}^2(\eta_1+\eta_2)^2-N_{13}^2N_{23}^2]}{[N_{13}^2N_{23}^2-N_{12}^2(\eta_1-\eta_2)^2]}. \tag{5}$$

Substituting the solutions to equations (3), namely

$$2N_{ij}^2=(1-\eta_1^2-\eta_2^2-\eta_3^2)+2\eta_k^2 \qquad (i,j,k)=(1,2,3) \tag{6}$$

into equation (5), we have, after much algebra,

$$\tan^2\beta_{12}=AB/CD \tag{7}$$

where

$$A=(\eta_1+\eta_2+\eta_3)^2-1 \tag{8}$$

$$B=1-(\eta_1+\eta_2-\eta_3)^2 \tag{9}$$

$$C=1-(\eta_2+\eta_3-\eta_1)^2 \tag{10}$$

$$D=1-(\eta_1+\eta_3-\eta_2)^2. \tag{11}$$

Similarly, for the other β we have

$$\tan^2\beta_{13}=AD/BC \tag{12}$$

and

$$\tan^2\beta_{23}=AC/BD. \tag{13}$$

Thus, we have obtained the appropriate relations for the β in terms of the η. We now consider the inverse relationships.

From the expressions for A, B, C and D, we have

$$2\eta_1=(1-B)^{1/2}+(1-D)^{1/2} \tag{14}$$

$$2\eta_2=(1-B)^{1/2}+(1-C)^{1/2} \tag{15}$$

$$2\eta_3=(1-C)^{1/2}+(1-D)^{1/2} \tag{16}$$

$$(A+1)^{1/2}=(1-B)^{1/2}+(1-C)^{1/2}+(1-D)^{1/2}. \tag{17}$$

The latter expression can be written in the more convenient form, by dividing by $A^{1/2}$ and writing t for $1/A$,

$$(1+t)^{1/2}=(t-b)^{1/2}+(t-c)^{1/2}+(t-d)^{1/2} \tag{18}$$

where

$$b=B/A=\cot\beta_{13}\cot\beta_{23} \tag{19}$$

$$c=C/A=\cot\beta_{12}\cot\beta_{13} \tag{20}$$

$$d=D/A=\cot\beta_{12}\cot\beta_{23}. \tag{21}$$

The solution of equation (18) is

$$t=\frac{[(1+b+c+d)^2-4(bc+cd+db)]^2+64bcd}{8(1+b+c-d)(1+b-c+d)(1-b+c+d)}. \tag{22}$$

From this value of t, we calculate the values of the η from

$$\eta_1=[(t-b)^{1/2}+(t-d)^{1/2}]/2t^{1/2} \tag{23}$$

$$\eta_2=[(t-b)^{1/2}+(t-c)^{1/2}]/2t^{1/2} \tag{24}$$

$$\eta_3=[(t-c)^{1/2}+(t-d)^{1/2}]/2t^{1/2}. \tag{25}$$

3. Application

Using the numerical method given in our recent paper (Kabir and Kermode 1987), we calculated the S matrix for a three coupled channel scattering process numerically. The diagonal amplitudes and the off-diagonal phases of the 3×3 unitary S matrix were calculated using equations (1) and (2). These values were used in equations (7), (12) and (13), and (23)-(25), and found to be correct. Graphs of the off-diagonal phases and the diagonal amplitudes as a function of energy, for a square well potential with one bound state, are plotted in figure 1.

4. Conclusion

The unitary symmetric 3×3 S matrix requires six real numbers to specify it uniquely. If three of those are taken to be the phases of the diagonal elements (i.e. twice the real phase shifts), the other three can be taken from the set of (a) diagonal amplitudes (η_i), (b) off-diagonal phases (β_{ij}), (c) off-diagonal amplitudes (N_{ij}) or (d) η_1, η_2 and N_{12}.

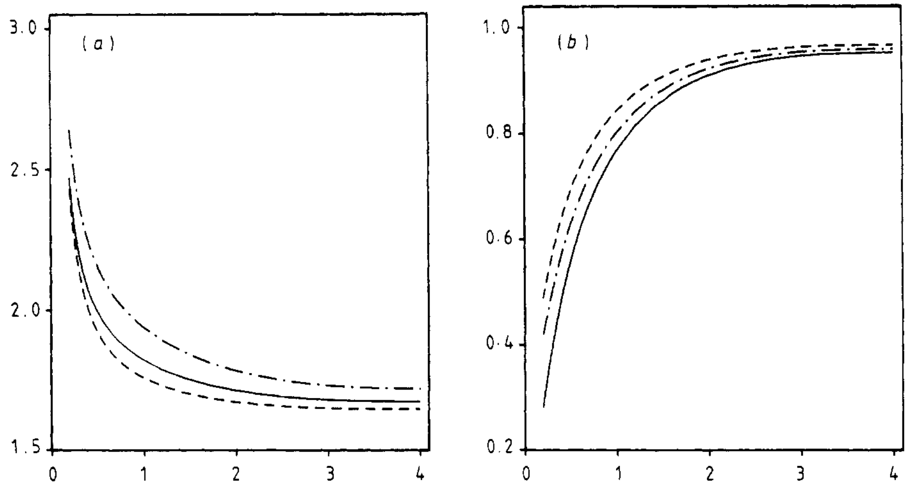

Figure 1. The energy dependence (in fm^{-2}) of the diagonal amplitudes and the off-diagonal phases, for a three coupled channel square well potential V_{ij}, where $V_{1j} = (0.33, 0.23, 0.27)$ fm^{-2}, $V_{2j} = (0.23, 0.30, 0.19)$ fm^{-2} and $V_{3j} = (0.27, 0.19, 0.28)$ fm^{-2}, and range 2 fm, which supports a bound state. (*a*) shows β_{12} (full curve), β_{13} (broken curve) and β_{23} (chain curve) in radians. (*b*) shows η_1 (full curve), η_2 (broken curve) and η_3 (chain curve).

The procedure for obtaining the complete S matrix for each of these cases is as follows.

Case (a). Calculate the amplitudes N_{ij} from the solutions to equation (3), i.e.

$$N_{12}^2 = (1 - \eta_1^2 - \eta_2^2 + \eta_3^2)/2 \tag{26}$$

and cyclic. Calculate the phases β_{ij} from equations (7)–(13).

Case (b). Calculate the amplitudes η_j from equations (19)–(25) and the amplitudes N_{ij} from equation (26).

Case (c). Calculate the amplitudes η_i from

$$\eta_1^2 = 1 - N_{12}^2 - N_{13}^2 \tag{27}$$

and cyclic, which is equation (3). The phases β_{ij} follow from equations (7)–(13).

Case (d). Calculate

$$N_{13}^2 = 1 - \eta_1^2 - N_{12}^2 \tag{28}$$

$$N_{23}^2 = 1 - \eta_2^2 - N_{12}^2 \tag{29}$$

$$\eta_3^2 = \eta_1^2 + \eta_2^2 + 2N_{12}^2 - 1 \tag{30}$$

and then the phases β_{ij} as above.

This may be particularly useful when the inelastic 2×2 submatrix S_e is considered (Kabir and Kermode 1987). Five parameters to determine S_e are δ_1, δ_2, η_1, η_2 and N_{12}. However, we note that the phase shifts δ_1 and δ_2 are *not* the two (bar) phase shifts in the usual expression for S_e. Without a consideration of the 4×4 matrix, it is not possible to see how the sixth parameter in the general parametrisation of S_e is not free in the three-channel case.

The relations that we have presented in this paper will prove useful in checking the numerical solutions of the Schrödinger equation in the case of three open channels. Also, in the construction of the S matrix from experimental data, the formulae (7)–(13)

and (23)-(25) will provide information as to whether the physical process involves three channels (the formulae are valid) or four or more channels (the formulae are not valid).

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Note added in proof. Dr S Klarsfeld has drawn our attention to two interesting papers on three-channel systems by Waldenstrom (1974, 1981). These papers give unitary bounds on the free parameters but do not contain the relations reported in our present paper. We thank Dr Klarsfeld for this information.

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